

Superradiant scattering of electromagnetic waves emitted from disk around Kerr black holes

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We study electromagnetic perturbations around a Kerr black hole surrounded by a thin disk on the equatorial plane. Our main purpose is to reveal the black hole superradiance of electromagnetic waves emitted from the disk surface. The outgoing Kerr-Schild field is used to describe the disk emission, and the superradiant scattering is represented by a vacuum wave field which is added to satisfy the ingoing condition on the horizon. The formula to calculate the energy flux on the disk surface is presented, and the energy transport in the disk-black hole system is investigated. Within the low-frequency approximation we find that the energy extracted from the rotating black hole is mainly transported back to the disk, and the energy spectrum of electromagnetic waves observed at infinity is also discussed.

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I. INTRODUCTION

It is widely believed that there exists a rotating black hole surrounded by a disk in the central region of highly energetic astrophysical objects, such as active galactic nuclei, x-ray binary system and gamma-ray burst sources. In the disk-black hole system, the propagation of electromagnetic waves emitted from the disk surface will be strongly affected by the gravity of the central black hole. For example, the black hole shadow is expected to be an important phenomenon from which the black hole spin can be observationally estimated [1–3]. The basic analysis of the shadow profile is relied on the ray-tracing method, and some features depending on various disk models have been investigated in [4, 5]. The further developments have been attempted in recent papers which take into account the effects of plasma and wave scattering [6, 7]. Another key aspect of wave propagation in a rotating black hole spacetime will be the effect of superradiance. As was shown in [8, 9], the amplitude of incident waves propagating to the horizon can be amplified through the scattering process to extract the rotational energy of a black hole. Though this efficiency has been calculated for waves incident from infinitely distant regions, in the disk-black hole system it is important to consider the electromagnetic waves which should occur on the disk surface. In this paper we would like to assume a thin disk and deal with the problem of propagation of waves incident from the equatorial plane in the context of the superradiant scattering rather than from the viewpoint of the black hole shadow.

The electromagnetic waves emitted from the disk surface should be directly carried away to infinity, and partly absorbed by the black hole. Our main interest is focused on the electromagnetic energy transport in the disk-black hole system. We will clarify how the wave absorption across the horizon generates superradiant energy outflows from the black hole to disk and infinity. This energy transport to the disk may be interpreted as a feedback mechanism which plays a role of disk reheating, while the contribution to the energy flux at infinity may be useful for an observational check of the superradiance effect. (Such energy outflow may also contributes to a change of the black hole shadow. However, to pursue this possibility is beyond the scope of this paper.)

Because the ray-tracing method is not applicable to the analysis of the superradiant scattering of electromagnetic waves, our approach to the problem starts from solving the vacuum Maxwell equations in Kerr geometry. The assumed boundary condition is the existence of a thin disk (i.e., a surface current) on the equatorial plane. Namely, it is required that some components of electromagnetic fields become discontinuous at the equatorial plane, and outgoing energy fluxes are emitted from both the upper and lower sides of the disk surface.

The Kerr-Schild (K-S) formalism for solving the Einstein-Maxwell equations is a useful method to overcome the mathematical difficulty due to the disk boundary contribution [10, 11]. In fact, if the electromagnetic fields are treated as perturbations in Kerr geometry, all the field components are simply derived by two arbitrary complex functions [12], which can be appropriately chosen according to the disk boundary condition. Hence, in Sec. II, we introduce the outgoing K-S field $F_{\mu\nu}^{\text{KS}}$ as a model of the disk emission and discuss a singular behavior of the two complex functions

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at the equatorial plane. Unfortunately, the outgoing K-S field fails to satisfy the condition that no outgoing waves are present on the horizon. Hence, the physical field satisfying the horizon boundary condition should be modified to the form $F_{\mu\nu} = F_{\mu\nu}^{\text{KS}} + F_{\mu\nu}^{\text{SW}}$, where the additional vacuum field $F_{\mu\nu}^{\text{SW}}$ be interpreted as the non-Kerr part due to the superradiant scattering of waves emitted from the disk and can be continuous even at the equatorial plane. In order to facilitate the analysis of the scattering problem, we consider the Newman-Penrose quantities [9, 13] corresponding to the electromagnetic field $F_{\mu\nu} = F_{\mu\nu}^{\text{KS}} + F_{\mu\nu}^{\text{SW}}$, and in Sec. III we express them as the infinite sums of a complete set of modes. Based on the mode decomposition, in Sec. IV, we derive the formulae to calculate the energy fluxes at the boundary surfaces including the horizon, the equatorial disk and infinity. In Sec. V, the superradiant energy transport from the black hole to the disk and infinity is explicitly estimated within the low-frequency limit of the wave fields. Hereafter we use units such that $c = G = 1$.

II. KERR-SCHILD FIELD AND NEWMAN-PENROSE QUANTITIES

Let us consider the electromagnetic waves emitted from disk surface around Kerr black hole, using the framework of the Kerr-Schild formalism (see details in [10]). Though this formalism is introduced to solve the full Einstein-Maxwell equations, it may be applied to obtain electromagnetic perturbations on Kerr background. The metrical ansatz is

$$g^{\mu\nu} = \eta^{\mu\nu} - 2He^{3\mu}e^{3\nu}, \quad (1)$$

where $\eta^{\mu\nu}$ is the metric of an auxiliary Minkowski spacetime, H is scalar function, and $e^{3\mu}$ is a null vector field, which is tangent to a geodesic and shear-free principal null congruence. It is convenient to calculate the Einstein-Maxwell equations using tetrad components. All other null tetrad vectors are defined by the condition

$$g_{ab} = e_a^\mu e_{b\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = g^{ab}, \quad (2)$$

where latin and greek suffixes mean tetrad and tensor suffixes, respectively. A tensor $T_\mu^{\nu\dots}$ is related to its tetrad components $T_a^{b\dots}$ by either of the two equivalent relations

$$T_a^{b\dots} = e_a^\mu e_\nu^b \dots T_\mu^{\nu\dots}, \quad T_\mu^{\nu\dots} = e_\mu^a e_b^\nu \dots T_a^{b\dots}. \quad (3)$$

The essential point of the Kerr-Schild formalism is to use the complex form of electromagnetic field tensors given by

$$\mathcal{F}_{\mu\nu} \equiv F_{\mu\nu} + \frac{1}{2}i\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \quad (4)$$

where $\epsilon_{\mu\nu\rho\sigma}$ is completely skew-symmetric, and equal to $\epsilon_{1234} = (-g)^{1/2}$. The corresponding null tetrad components are

$$\mathcal{F}_{ab} = F_{ab} + \frac{1}{2}i\epsilon_{abcd}F^{cd}, \quad (5)$$

where ϵ_{abcd} is completely skew-symmetric, and $\epsilon_{1234} = i$. By virtue of the definition (5) and the Einstein equations, the tetrad components \mathcal{F}_{32} , \mathcal{F}_{41} , and \mathcal{F}_{42} are found to be zero. The electromagnetic fields are completely determined by only two complex components \mathcal{F}_{12} , and \mathcal{F}_{31} . It is interesting to note that the Kerr-Schild form remains valid, even if a back reaction on the gravitational field by the electromagnetic field is considered.

A part of the Maxwell equations allows to write the tetrad components as

$$\mathcal{F}_{12} = \mathcal{F}_{34} = AZ^2, \quad (6)$$

$$\mathcal{F}_{31} = \gamma Z - (AZ)_{,1}, \quad (7)$$

where Z is the complex expansion of the null vector e^3 , and commas denote the directional derivatives along chosen null tetrad vectors. The functions A and γ should be determined by solving the other Maxwell equations.

In this paper we treat the electromagnetic fields as perturbations on Kerr background, and use the outgoing Kerr-Schild coordinate system to describe electromagnetic waves emitted from disk to infinity. The outgoing Kerr-Schild

form of the Kerr metric is given by

$$\begin{aligned} ds^2 = & -d\tilde{t}^2 + dr^2 + \Sigma d\theta^2 + (r^2 + a^2) \sin^2 \theta d\tilde{\varphi}^2 \\ & + 2a \sin^2 \theta dr d\tilde{\varphi} \\ & + \frac{2Mr}{\Sigma} (d\tilde{t} - dr - a \sin^2 \theta d\tilde{\varphi})^2, \end{aligned} \quad (8)$$

where $\Sigma \equiv r^2 + a^2 \cos^2 \theta$, and M and a denote the mass and the angular momentum per unit mass of the black hole, respectively. The function H in Eq. (1) is given by $2Mr/P^2\Sigma$, where $P = 1/\sqrt{2} \cos^2(\theta/2)$, and the null tetrad vectors are given by

$$e^1 = 2^{-\frac{1}{2}} e^{i\tilde{\varphi}} [-\tan(\theta/2), \tan(\theta/2), (r - ia \cos \theta), (a + ir) \sin \theta], \quad (9a)$$

$$e^2 = 2^{-\frac{1}{2}} e^{-i\tilde{\varphi}} [-\tan(\theta/2), \tan(\theta/2), (r + ia \cos \theta), (a - ir) \sin \theta], \quad (9b)$$

$$e^3 = P [-1, 1, 0, a \sin^2 \theta], \quad (9c)$$

$$e^4 = 2^{-\frac{1}{2}} [1, \cos \theta, -r \sin \theta, 0] + H e^3, \quad (9d)$$

with e^3 an outgoing null geodesic. Then, the functions A and γ for electromagnetic perturbations on Kerr background can be written as

$$A = \frac{\psi(Y, \tau)}{P^2}, \quad (10)$$

$$\gamma = \frac{2^{1/2} \psi_{, \tau}}{P^2 Y} + \frac{\phi(Y, \tau)}{P}, \quad (11)$$

where $\tau = \tilde{t} - r + ia \cos \theta$ because e^3 is chosen as an outgoing vector field, $Y = e^{i\tilde{\varphi}} \tan(\theta/2)$ (see [12]), and comma means differentiation with respect to a given variable. Further, we obtain $Z/P = 1/(r - ia \cos \theta)$ for the complex expansion Z in Eqs. (6) and (7). It should be noted that the tetrad components \mathcal{F}_{ab} can be written by the two arbitrary complex functions $\psi(Y, \tau)$ and $\phi(Y, \tau)$ as follows,

$$\mathcal{F}_{12} = \frac{\psi}{(r - ia \cos \theta)^2}, \quad (12)$$

$$\begin{aligned} \mathcal{F}_{31} = & \frac{1}{r - ia \cos \theta} \left\{ e^{-i\tilde{\varphi}} \left[\frac{2 \cos^2(\theta/2)}{\tan(\theta/2)} \frac{r - ia}{r - ia \cos \theta} \psi_{, \tau} + \sin \theta \frac{r + ia}{(r - ia \cos \theta)^2} \psi \right] \right. \\ & \left. + \phi(Y, \tau) - \frac{\psi_{, Y}}{r - ia \cos \theta} \right\}. \end{aligned} \quad (13)$$

Hereafter we call this solution the Kerr-Schild field.

To see clearly superradiant energy transport in the disk-black hole system, it is convenient to introduce the Boyer-Lindquist coordinates, which lead to the metric

$$ds^2 = - \left(1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4aMr}{\Sigma} \sin^2 \theta dt d\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\mathcal{A}}{\Sigma} \sin^2 \theta d\varphi^2, \quad (14)$$

where $\Delta = r^2 + a^2 - 2Mr$, and $\mathcal{A} = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta$. The Boyer-Lindquist coordinates t and φ are related to the outgoing Kerr-Schild coordinates \tilde{t} and $\tilde{\varphi}$ as follows,

$$dt = d\tilde{t} + \frac{2Mr}{\Delta} dr, \quad d\varphi = d\tilde{\varphi} + \frac{a}{\Delta} dr. \quad (15)$$

If the electromagnetic perturbations $F_{\mu\nu}$ written in the Boyer-Lindquist coordinate system are assumed to be functions of three variables $\sigma t - \varphi$, r , θ only, we can simply expect that the superradiant scatterings occurs under the condition $\sigma < \Omega_H$ for the angular velocity Ω_H of the black hole and a frequency parameter σ [11]. The expectation motivates us to specify the complex function $\psi(Y, \tau)$ and $\phi(Y, \tau)$ in Eqs. (12) and (13) to the forms

$$\psi(Y, \tau) = \psi(X), \quad \phi(Y, \tau) \equiv (-ia\sigma X \psi_{, X} + \Psi(X))/Y. \quad (16)$$

where the complex variable X is defined by

$$X \equiv e^{-i\sigma\tau} Y = e^{-i\sigma\tau + i\tilde{\varphi}} \tan(\theta/2), \quad (17)$$

and the term $-ia\sigma X\psi_{,X}$ is included in ϕ to simplify the expression of $F_{\mu\nu}$ which will be given later. It is easy to see from Eqs. (12) and (13) that by virtue of the choice of ψ and ϕ the field components \mathcal{F}_{12} and \mathcal{F}_{31} depend on t and φ via the variable $\sigma\tau - \tilde{\varphi}$ in X , and we obtain

$$\sigma\tau - \tilde{\varphi} = \sigma t - \varphi + i\sigma a \cos\theta - \sigma r_* + \Omega_H L(r) \quad (18)$$

where the tortoise coordinate r_* is defined as

$$r_* \equiv \int \frac{r^2 + a^2}{\Delta} dr = r - r_1 + \frac{2Mr_1}{r_1 - r_2} \ln \left| \frac{r - r_1}{r_1 - r_2} \right| + \frac{2Mr_2}{r_2 - r_1} \ln \left| \frac{r - r_2}{r_1 - r_2} \right|, \quad (19)$$

using the outer and inner horizon radii r_1 and r_2 , respectively. For the function $L(r)$ given by

$$L(r) = \int \frac{2Mr_1}{\Delta} dr = \frac{2Mr_1}{r_1 - r_2} \ln \left| \frac{r - r_1}{r - r_2} \right|, \quad (20)$$

we can check the asymptotic behaviors such that $L - r_* \rightarrow 0$ on the outer horizon $r = r_1$, and $L \rightarrow 0$ at infinity $r \rightarrow \infty$.

It is a straight forward task to derive the field components in the Boyer-Lindquist coordinate system from \mathcal{F}_{12} and \mathcal{F}_{31} given by Eqs. (12) and (13). Using the specified form of ψ and ϕ , the K-S field components denoted by $F_{\mu\nu}^{\text{KS}}$ are given by

$$F_{tr}^{\text{KS}} = -\text{Re} \left[\frac{\psi(X)}{(r - ia \cos\theta)^2} - i \frac{a}{\Delta \sin\theta} (\Theta + i\Xi) \right], \quad (21a)$$

$$F_{t\theta}^{\text{KS}} = -\text{Re} \left[ia \sin\theta \frac{\psi(X)}{(r - ia \cos\theta)^2} + \frac{2}{\sin\theta} (\Theta + i\Xi) \right], \quad (21b)$$

$$F_{t\varphi}^{\text{KS}} = -\text{Re} \left[\frac{i}{\sin\theta} (\Theta + i\Xi) \right], \quad (21c)$$

$$F_{\theta\varphi}^{\text{KS}} = \text{Re} \left[-i(r^2 + a^2) \sin\theta \frac{\psi(X)}{(r - ia \cos\theta)^2} + 2a (\Theta + i\Xi) \right], \quad (21d)$$

$$F_{r\varphi}^{\text{KS}} = \text{Re} \left[-a \sin^2\theta \frac{\psi(X)}{(r - ia \cos\theta)^2} + i \frac{r^2 + a^2}{\Delta \sin\theta} (\Theta + i\Xi) \right], \quad (21e)$$

$$F_{r\theta}^{\text{KS}} = \frac{2\Sigma}{\Delta \sin\theta} \text{Re} \left[\frac{1}{\sin\theta} (\Theta + i\Xi) \right], \quad (21f)$$

where the function Θ and Ξ are defined as

$$\Theta \equiv \Psi(X) + X\psi_{,X} (2a\sigma \cos^2(\theta/2) - 1) / (r - ia \cos\theta), \quad (22)$$

$$\Xi \equiv X\psi_{,X} a\sigma [2r \cos^2(\theta/2) / (r - ia \cos\theta) - 1]. \quad (23)$$

Here we consider the boundary condition on the disk located at the equatorial plane $\theta = \pi/2$. It is well-known that any complex function which is not a constant should have a singularity on the complex plane. We will assume that the existence of a singularity in $\psi(X)$ and $\Psi(X)$ on the complex X -plane, is due to a surface current on the equatorial plane $\theta = \pi/2$. This means that the components $F_{t\theta}^{\text{KS}}$, $F_{\theta\varphi}^{\text{KS}}$, and $F_{r\theta}^{\text{KS}}$ (namely, the imaginary part of ψ and the real part of Θ) become discontinuous at $\theta = \pi/2$. Such a discontinuity will be generated if a branch point in ψ exists at $X = e^{i\beta}$ where β is a real constant. For example, as was discussed in [11], the function $\psi(X) = \psi_0 [(X^{-2} + X^2)^{3/2} - X^{-3} - X^3]^2$ with a real constant ψ_0 has four branch points at $X^2 = \pm i$, and the imaginary part of ψ and the real part of $X\psi_{,X}$ become discontinuous at $\theta = \pi/2$. In this case the ratio $\Psi(X)/X\psi_{,X}$ may be chosen to be a real constant, as will be done in (69). Further it should be noted that the absolute value $|X| = e^{a\sigma \cos\theta} \tan(\theta/2)$ becomes equal to unity at $\theta = \pi/2$. The branch point $X = e^{i\beta}$ may also appear on some conical plane $\theta = \theta_0 (\neq \pi/2)$ giving $|X| = 1$ if $a\sigma > 1$. Hence, in the following, the allowed range of the frequency parameter σ is limited to the range $0 < \sigma < 1/a$, for which we obtain $|X| < 1$ in the upper region $0 \leq \theta < \pi/2$ and $|X| > 1$ in the lower region $\pi/2 < \theta \leq \pi$.

We must also consider the regularity condition for $F_{\mu\nu}^{\text{KS}}$ at the polar axis (i.e., at $\theta = 0, \pi$). Noting that $|X| \simeq \sin(\theta/2)$ in the limit $\theta \rightarrow 0$ and $|X| \simeq 1/\cos(\theta/2)$ in the limit $\theta \rightarrow \pi$, we find the boundary condition for ψ and Ψ to be for $\theta \rightarrow 0$,

$$\psi(X) \sim \Psi(X) \sim X^2, \quad (24)$$

and for $\theta \rightarrow \pi$,

$$\psi(X) \sim \Psi(X) \sim 1/X^2. \quad (25)$$

Finally, let us discuss the boundary condition on the horizon and at infinity, by introducing the electromagnetic Newman-Penrose quantities ϕ_a ($a = 0, 1, 2$) [13]. Using the Kinnersley's tetrad well-behaved on the past horizon such that

$$l^\mu = [(r^2 + a^2)/\Delta, 1, 0, a/\Delta], \quad (26a)$$

$$n^\mu = [r^2 + a^2, -\Delta, 0, a]/2\Sigma, \quad (26b)$$

$$m^\mu = [ia \sin \theta, 0, 1, i/\sin \theta]/2^{1/2}(r + ia \cos \theta), \quad (26c)$$

the Newman-Penrose quantities are defined as

$$\phi_0 = F_{\mu\nu} l^\mu m^\nu, \quad (27a)$$

$$\phi_1 = \frac{1}{2} F_{\mu\nu} (l^\mu n^\nu + \bar{m}^\mu m^\nu), \quad (27b)$$

$$\phi_2 = F_{\mu\nu} \bar{m}^\mu n^\nu, \quad (27c)$$

and satisfy the Maxwell equations written by

$$\frac{(r - ia \cos \theta)}{\sqrt{2}} \left(\mathcal{L}_1 - \frac{ia \sin \theta}{(r - ia \cos \theta)} \right) \phi_0 = \mathcal{D}_0 [\phi_1 (r - ia \cos \theta)^2] \quad (28a)$$

$$\frac{1}{\sqrt{2}(r - ia \cos \theta)^2} \mathcal{L}_0 [\phi_1 (r - ia \cos \theta)^2] = \mathcal{D}_0 [\phi_2 (r - ia \cos \theta)], \quad (28b)$$

$$-\frac{\Delta}{\sqrt{2}(r - ia \cos \theta)^2} \mathcal{D}_0^\dagger [\phi_1 (r - ia \cos \theta)^2] = \mathcal{L}_1^\dagger [\phi_2 (r - ia \cos \theta)], \quad (28c)$$

$$-\frac{(r - ia \cos \theta)\Delta}{\sqrt{2}} \left(\mathcal{D}_1^\dagger - \frac{1}{(r - ia \cos \theta)} \right) \phi_0 = \mathcal{L}_0^\dagger [\phi_1 (r - ia \cos \theta)^2], \quad (28d)$$

where the differential operators \mathcal{D}_n and \mathcal{L}_n are defined by

$$\mathcal{D}_n \equiv l^\mu \partial_\mu + 2n(r - M)/\Delta, \quad (29a)$$

$$\mathcal{D}_n^\dagger \equiv -2(\Sigma/\Delta)n^\mu \partial_\mu + 2n(r - M)/\Delta, \quad (29b)$$

$$\mathcal{L}_n \equiv \sqrt{2}(r - ia \cos \theta)\bar{m}^\mu \partial_\mu + n \cot \theta, \quad (29c)$$

$$\mathcal{L}_n^\dagger \equiv \sqrt{2}(r + ia \cos \theta)m^\mu \partial_\mu + n \cot \theta. \quad (29d)$$

From Eqs. (21), (26) and (27) we obtain the Newman-Penrose quantities for the Kerr-Schild field $F_{\mu\nu}^{\text{KS}}$ as follows,

$$\phi_0^{\text{KS}} = 0, \quad (30a)$$

$$\phi_1^{\text{KS}} = \frac{\psi(X)}{2(r - ia \cos \theta)^2}, \quad (30b)$$

$$\phi_2^{\text{KS}} = \frac{1}{\sqrt{2}(r - ia \cos \theta) \sin \theta} \left[\Psi(X) + \frac{X\psi, X}{(r - ia \cos \theta)} (i\sigma r \cos \theta + a\sigma - 1) \right]. \quad (30c)$$

Because the electromagnetic waves are assumed to be emitted from the disk, no ingoing waves should exist at infinity. Hence, we require the asymptotic behavior of the Newman-Penrose quantities obeying Eqs. (28) to be

$$\phi_0 \sim 1/r^3, \quad \phi_2 \simeq (1/r)g(t - r_*), \quad (31)$$

in the limit $r \rightarrow \infty$. On the other hand no outgoing waves should not exist on the horizon, and we require

$$\phi_0 \simeq (1/\Delta)f(t + r_*), \quad \phi_2 \sim \Delta, \quad (32)$$

in the limit $\Delta \rightarrow 0$. It is easy to see that the Kerr-Schild field satisfies the boundary condition only at infinity. The horizon boundary condition breaks down, because ϕ_2^{KS} does not vanish at $\Delta = 0$. Therefore, to obtain the physical field ϕ_a which is well-behaved on the horizon, some vacuum field denoted by ϕ_a^{SW} is added to the Kerr-Schild field ϕ_a^{KS} as follows

$$\phi_0 = \phi_0^{\text{SW}}, \quad \phi_1 = \phi_1^{\text{KS}} + \phi_1^{\text{SW}}, \quad \phi_2 = \phi_2^{\text{KS}} + \phi_2^{\text{SW}}, \quad (33)$$

where ϕ_2 is required to vanish on the horizon. Though the Kerr-Schild field describes the disk emission, the additional field is expected to represent the effect of wave scattering (or absorption) by the black hole. In the next section we will describe the scheme to obtain the additional field ϕ_a^{SW} , by imposing the conditions (31) and (32) on ϕ_a .

III. WAVE SCATTERING

As the first step to analyze the scattered wave field ϕ_a^{SW} , let us expand the functions $\psi(X)$ and $\Psi(X)$ in Eqs. (30) as

$$\psi(X) = \sum_m a_m X^m, \quad (34)$$

$$\Psi(X) = \sum_m b_m X^m, \quad (35)$$

where from the condition (24) m runs from 2 to ∞ for $0 < |X| < 1$ (corresponding to the upper region $0 < \theta < \pi/2$), while from the condition (25) it runs from -2 to $-\infty$ for $1 < |X| < \infty$ (corresponding to the lower region $\pi/2 < \theta < \pi$). Such an expansion will be possible, because $\psi(X)$ and $\Psi(X)$ are assumed to be regular except at branch points on the equatorial plane $|X| = 1$. Note that the m -th terms ψ_m and Ψ_m are proportional to $\exp[im(\sigma t - \varphi)]$, which represents a mode with the wave frequency

$$\omega_m \equiv m\sigma, \quad (36)$$

for $m > 0$, while the wave frequency should be understood to be $-\omega_m$ for $m < 0$. By virtue of the expansion of ψ and Ψ , the Kerr-Schild field ϕ_2^{KS} is rewritten in the form

$$(r - ia \cos \theta)^2 \phi_2^{\text{KS}} = \sum_{m=-\infty}^{\infty} e^{-i\omega_m t + im\varphi} S_m^{\text{KS}}(r, \theta) e^{i(\omega_m r_* - m\Omega_H L)}, \quad (37)$$

$$S_m^{\text{KS}}(r, \theta) = H_m \frac{e^{a\omega_m \cos \theta} \tan^m(\theta/2)}{\sqrt{2} \sin \theta} [(r - ia \cos \theta)b_m + a_m (i\omega_m r \cos \theta + a\omega_m - m)]. \quad (38)$$

Because the modes for $m = \pm 1, 0$ should not be included in Eq.(37), we have $H_{\pm 1} = H_0 = 0$. Further the expansion forms (34) and (35) mean that for $m \geq 2$ (or $m \leq -2$) ψ and Ψ must vanish in the lower (or upper) region. Hence, the factor H_m in (38) is given by the step function such that $H_m = (1 + m/|m|)/2$ in the range $0 \leq \theta < \pi/2$ and $H_m = (1 - m/|m|)/2$ in the range $\pi/2 < \theta \leq \pi$. From Eq. (37) we have the asymptotic behavior near the horizon $r = r_1$ as follows,

$$(r - ia \cos \theta)^2 \phi_2^{\text{KS}} \simeq \sum_m e^{-i\omega_m t + im\varphi} S_m^{\text{KS}}(r_1, \theta) e^{ik_m r_*}, \quad k_m \equiv \omega_m - m\Omega_H \quad (39)$$

which should be canceled out by the scattered-wave field ϕ_2^{SW} according to the horizon boundary condition. The easier way to construct such a vacuum non-Kerr-Schild field will be to use the expansion form written by the spin-weighted angular functions $S_{lm}^{(\pm 1)}(\theta)$. The application of this mode decomposition to ϕ_0^{SW} and ϕ_2^{SW} leads to the result

$$\phi_0^{\text{SW}} = \sum_{m,l} e^{-i\omega_m t + im\varphi} S_{lm}^{(1)}(\theta) R_{lm}^{(1)}(r), \quad (40)$$

$$(r - ia \cos \theta)^2 \phi_2^{\text{SW}} = \sum_{m,l} e^{-i\omega_m t + im\varphi} S_{lm}^{(-1)}(\theta) R_{lm}^{(-1)}(r), \quad (41)$$

where $R_{lm}^{(s)}$ is the radial function and $l \geq |m| \geq 2$. Note that the component ϕ_1^{SW} can be derived by using the Maxwell equations (28). Therefore we consider hereafter only the two components ϕ_0^{SW} and ϕ_2^{SW} .

The asymptotic behaviors of $R_{lm}^{\pm 1}$ near the horizon and at infinity are well-known. For example, in the limit $r \rightarrow r_1$, we give the radial function as follows

$$R_{lm}^{(1)} \simeq C_{lm}^{\text{In}} \Delta^{-1} e^{-ik_m r_*} + C_{lm}^{\text{Out}} e^{ik_m r_*}, \quad (42)$$

$$R_{lm}^{(-1)} \simeq D_{lm}^{\text{In}} \Delta e^{-ik_m r_*} + D_{lm}^{\text{Out}} e^{ik_m r_*}, \quad (43)$$

where the outgoing parts with the amplitudes C_{lm}^{Out} and D_{lm}^{Out} are also included to satisfy the condition $\phi_2^{\text{SW}} + \phi_2^{\text{KS}} \rightarrow 0$ on the horizon. On the other hand, in the limit $r \rightarrow \infty$ where no incoming waves exist we obtain

$$R_{lm}^{(1)} \simeq E_{lm}^{\text{Out}} e^{-i\omega_m r_*} / r^3, \quad (44)$$

$$R_{lm}^{(-1)} \simeq F_{lm}^{\text{Out}} r e^{-i\omega_m r_*}. \quad (45)$$

The coefficient ratios $C_{lm}^{\text{In}}/D_{lm}^{\text{In}}$ and $E_{lm}^{\text{Out}}/F_{lm}^{\text{Out}}$ have been derived in [9] using the Teukolsky equations, and the results are given by

$$C_{lm}^{\text{In}}/D_{lm}^{\text{In}} = -\frac{32ik_m M^2 r_1^2 (-ik_m + 2\epsilon)}{B}, \quad (46)$$

$$E_{lm}^{\text{Out}}/F_{lm}^{\text{Out}} = -\frac{B}{2\omega_m^2}, \quad (47)$$

where $B = (E + a^2\omega_m^2 - 2a\omega_m m)^2 + 4ma\omega_m - 4a^2\omega_m^2$ and $\epsilon = (r_1 - r_2)/4Mr_1$, and E is the eigenvalue of the angular equation. In the low-frequency limit $a\omega_m \rightarrow 0$, we have $E \rightarrow l(l+1)$, which is the case analyzed in Sec. V. Further, we obtain the ratio $C_{lm}^{\text{Out}}/D_{lm}^{\text{Out}}$ written as

$$C_{lm}^{\text{Out}}/D_{lm}^{\text{Out}} = \frac{B}{8ik_m M^2 r_1^2 (ik_m + 2\epsilon)}. \quad (48)$$

From Eqs. (39) and (43), the asymptotic behavior of ϕ_2 near the horizon is written as

$$(r - ia \cos \theta) \phi_2 \simeq \sum_{m,l} e^{-i\omega_m t + im\varphi} \left[S_{lm}^{(-1)}(\theta) (D_{lm}^{\text{In}} \Delta e^{-ik_m r_*} + D_{lm}^{\text{Out}} e^{ik_m r_*}) + S_m^{\text{KS}}(r_1, \theta) e^{ik_m r_*} \right] \quad (49)$$

Hence, the horizon boundary condition for ϕ_2 leads to the relation

$$\sum_{l \geq |m|} D_{lm}^{\text{Out}} S_{lm}^{(-1)}(\theta) = -S_m^{\text{KS}}(r_1, \theta), \quad (50)$$

from which the coefficient D_{lm}^{Out} is determined by

$$D_{lm}^{\text{Out}} \equiv - \int_0^\pi S_m^{\text{KS}}(r_1, \theta) S_{lm}^{(-1)}(\theta) \sin \theta d\theta. \quad (51)$$

for the given Kerr-Schild field.

The important problem to be solved in relation to the superradiant scattering of disk emission is to estimate the ratios $C_{lm}^{\text{In}}/D_{lm}^{\text{Out}}$ and $F_{lm}^{\text{Out}}/D_{lm}^{\text{Out}}$, based on the radial equation [9]

$$\Delta \frac{d^2 R^{(-1)}}{dr^2} + \left[\frac{K^2 + 2i(r-M)K}{\Delta} - 4ir\omega_m - \lambda \right] R^{(-1)} = 0, \quad (52)$$

where $K \equiv (r^2 + a^2)\omega_m - am$, λ is separation constant written by $\lambda = E - 2am\omega_m + a^2\omega_m^2$.

Finally, we summarize our proposed approach which is the derivation of the scattered wave $F_{\mu\nu}^{\text{SW}}$. In our approach, the property of the disk emission is given by the Kerr-Schild field ϕ_a^{KS} , that is, the two any complex functions ψ and Ψ or the expansion coefficients a_m and b_m in Eqs. (34) and (35). However the horizon boundary condition (32) breaks down, because ϕ_2^{KS} dose not vanish at $\Delta = 0$. Therefore, to obtain the physical field ϕ_a which is well-behaved on the horizon, we introduced the vacuum field ϕ_a^{SW} which represents the effect of wave scattering (or absorption) by the black hole. Then, the coefficient D_{lm}^{Out} in the outgoing part of the scattered wave should be determined by Eq. (51), using the Kerr-Schild field ϕ_a^{KS} . To determine the scattered field ϕ_a^{SW} , we must solve the radial equation (52), using the boundary value given by Eq. (51) on the horizon. If the radial equation (52) is solved, the ratios $D_{lm}^{\text{In}}/D_{lm}^{\text{Out}}$ and $F_{lm}^{\text{Out}}/D_{lm}^{\text{Out}}$ will be obtained. Therefore, the scattered wave ϕ_a^{SW} is represented only by the coefficients D_{lm}^{Out} connected with Kerr-Schild field on the horizon.

Before pursuing the analysis of Eq. (52) in details, we must present the formulae to calculate the energy fluxes from the disk, on the horizon and at infinity, because our main purpose is to clarify the energy transport via the superradiant scattering process in the disk-black hole system. This will be done in the next section.

IV. ENERGY FLUX

Using the Newman-Penrose quantities obtained in the previous section, let us present the useful expressions of the energy flux vector defined by

$$\mathcal{E}^\mu \equiv -T^\mu_t, \quad (53)$$

where

$$T_{\mu\nu} = \frac{1}{4\pi} [\phi_0 \bar{\phi}_0 n_\mu n_\nu + \phi_2 \bar{\phi}_2 l_\mu l_\nu + 2\phi_1 \bar{\phi}_1 (l_{(i} n_{j)} + m_{(i} \bar{m}_{j)}) - 4\bar{\phi}_0 \phi_1 n_{(i} m_{j)} - 4\bar{\phi}_1 \phi_2 l_{(i} m_{j)} + 2\phi_2 \bar{\phi}_0 m_i m_j] + \text{C.C.} \quad (54)$$

Because the component ϕ_1 given by ϕ_0 and ϕ_2 through the Maxwell equations (28), the energy flux vector \mathcal{E}^μ can be written by ϕ_0 and ϕ_2 only.

Note that the energy flux vector for the wave fields considered here is oscillatory with respect to the time t (as well as the azimuthal angle φ). To estimate the efficiency of the energy transport, we must consider the time-average quantities such that

$$\langle A \rangle \equiv \frac{\sigma}{2\pi} \int_0^{2\pi/\sigma} A dt, \quad (55)$$

with the frequency parameter σ . Because it is easy to see that the time-dependence on the energy flux vectors \mathcal{E}^μ arises from the electromagnetic field components ϕ_a , we consider the time-averaged quantities $\langle \phi_a \bar{\phi}_b \rangle$ written as

$$\langle \phi_a \bar{\phi}_b \rangle = \frac{\sigma}{2\pi} \int_0^{2\pi/\sigma} \phi_a \bar{\phi}_b dt. \quad (56)$$

Note that the electromagnetic field components ϕ_a are expanded as

$$\phi_a = \sum_{m=-\infty}^{\infty} \phi_{am}, \quad (57)$$

with $|m| \geq 2$. In particular, ϕ_{0m} and ϕ_{2m} are written as

$$\phi_{0m} = e^{-i\omega_m t + im\varphi} \sum_{l \geq |m|} S_{lm}^{(1)}(\theta) R_{lm}^{(1)}(r), \quad (58)$$

$$(r - ia \cos \theta)^2 \phi_{2m} = e^{-i\omega_m t + im\varphi} \left[\sum_{l \geq |m|} S_{lm}^{(-1)}(\theta) R_{lm}^{(-1)}(r) + S_m^{\text{KS}}(r, \theta) e^{im(\sigma r_* - \Omega_H L(r))} \right]. \quad (59)$$

From Eqs. (58) and (59), it is easy to see that the time-averaged quantities $\langle \phi_a \bar{\phi}_b \rangle$ are given by the mode decomposition as follows,

$$\langle \phi_a \bar{\phi}_b \rangle = \sum_{m=-\infty}^{\infty} \phi_{am} \bar{\phi}_{bm}, \quad (60)$$

with $|m| \geq 2$, which will lead to the time-averaged energy flux vectors written as

$$\langle \mathcal{E}^\mu \rangle = \sum_{m=-\infty}^{\infty} \langle \mathcal{E}_m^\mu \rangle, \quad (61)$$

with $|m| \geq 2$. Hereafter, we consider the mode-decomposed and time-averaged energy flux vector $\langle \mathcal{E}_m^\mu \rangle$.

We consider the angular component $\langle \mathcal{E}^\theta \rangle$ of the energy flux vector as the emission from the disk surface. Nothing that $\sqrt{\Sigma} = r$ at $\theta = \pi/2$, the energy flux \mathcal{E}_D per unit area can be evaluated as

$$\mathcal{E}_m^D(r) = r \langle \mathcal{E}_m^+ \rangle + r \langle \mathcal{E}_m^- \rangle, \quad (62)$$

where $\langle \mathcal{E}_m^\pm \rangle$ are equal to $\pm \langle \mathcal{E}_m^\theta \rangle$ in the limit $\theta \rightarrow \pi/2 \pm 0$ corresponding to the disk emission from the upper and lower side, respectively. Then, $\langle \mathcal{E}_m^\pm \rangle$ are obtained as

$$\langle \mathcal{E}_m^\pm \rangle = -\frac{1}{8\pi r^2(m - a\omega_m)} \partial_r A_m^\pm - \frac{\omega_m}{4\pi(m - a\omega_m)} i B_m^\pm, \quad (63)$$

$$A_m^\pm = \left\{ \Delta(\bar{\phi}_{0m}^\pm \phi_{2m}^\pm + \phi_{0m}^\pm \bar{\phi}_{2m}^\pm) + 2r^2 \phi_{2m}^\pm \bar{\phi}_{2m}^\pm + \frac{\Delta^2}{2r^2} \phi_{0m}^\pm \bar{\phi}_{0m}^\pm \right\}, \quad (64)$$

$$B_m^\pm = (\bar{\phi}_{0m}^\pm \phi_{2m}^\pm - \phi_{0m}^\pm \bar{\phi}_{2m}^\pm). \quad (65)$$

To evaluate the total flux radiated from disk surface, it is easy to obtain the total energy flux E_m^D as follows

$$E_m^D \equiv 2\pi \int_{r_1}^{\infty} \mathcal{E}_m^D r dr. \quad (66)$$

After a tedious calculation, we have the total energy flux E_m^D given by

$$E_m^D = -\frac{1}{4(m - a\omega_m)} [A_m^+ - A_m^-]_{r_1}^{\infty} - \frac{\omega_m}{2(m - a\omega_m)} \int_{r_1}^{\infty} i(B_m^+ - B_m^-) r^2 dr. \quad (67)$$

Noting that for the Kerr-Schild field $\phi_0^{\text{KS}} = 0$ and $\phi_2^{\text{KS}} \rightarrow 0$ in the limit $\theta \rightarrow \pi/2 - 0$, the corresponding total energy flux $[E_m^D]^{\text{KS}}$ is obtained as

$$[E_m^D]^{\text{KS}} = \frac{1}{2(m - a\omega_m)} [(r^2 \phi_{2m}^{+\text{KS}} \bar{\phi}_{2m}^{+\text{KS}})_{\text{H}} - (r^2 \phi_{2m}^{+\text{KS}} \bar{\phi}_{2m}^{+\text{KS}})_{\infty}]. \quad (68)$$

To assume the continuity of ϕ_a^{SW} at $\theta = \pi/2$ on the horizon for $m \geq 2$, we impose the same continuity condition ϕ_a^{KS} which is interpreted as the condition that the disk does not extend to the horizon, and choose $(r^2 \phi_{2m}^{+\text{KS}} \bar{\phi}_{2m}^{+\text{KS}})_{\text{H}}$ to be zero. Then, from Eq. (39) it is easy to see that the coefficient b_m is determined as follows,

$$b_m = \frac{m - a\omega_m}{r_1} a_m. \quad (69)$$

In particular, noting the continuity of the scattered field at $\theta = \pi/2$, namely, $\phi_a^{+\text{SW}} = \phi_a^{-\text{SW}}$, the total energy flux E_{Dm} is obtained as follows,

$$E_m^D = -\frac{(r^2 \phi_{2m}^{+\text{KS}} \bar{\phi}_{2m}^{+\text{KS}})_{\infty}}{2(m - a\omega_m)} - \frac{(r^2 \phi_{2m}^{+\text{KS}} \bar{\phi}_{2m}^{+\text{SW}})_{\infty}}{2(m - a\omega_m)} + \frac{\omega_m}{m - a\omega_m} \int_{r_1}^{\infty} \text{Im}(\phi_{2m}^{+\text{KS}} \phi_{0m}^{+\text{SW}}) r^2 dr. \quad (70)$$

For $m \leq -2$, we can use the same formula only by exchanging the subscript $+$ for $-$ and vice versa.

Next let us calculate the energy flux on the horizon, where we obtain the radial component $\langle \mathcal{E}_m^r \rangle$ of the energy flux vector as

$$\langle \mathcal{E}_m^r \rangle_{\text{H}} = -\frac{1}{8\pi k_m \Sigma_{\text{H}}} \left[\frac{\omega_m}{2Mr_1} (\Delta^2 \phi_{0m} \bar{\phi}_{0m})_{\text{H}} - \frac{\Omega_{\text{H}}}{2 \sin \theta} \partial_{\theta} \left(\frac{(\Delta^2 \phi_{0m} \bar{\phi}_{0m})_{\text{H}}}{\Sigma_{\text{H}}} \sin^2 \theta \right) \right]. \quad (71)$$

Further, we can evaluate the total flux E_{H} integrated over the whole horizon surface as follows,

$$E_m^{\text{H}} \equiv 2\pi \int_0^{\pi} \langle \mathcal{E}_m^r \rangle_{\text{H}} \Sigma_{\text{H}} \sin \theta d\theta, \quad (72)$$

which reduces to the form

$$E_m^{\text{H}} = -\frac{\omega_m}{8k_m Mr_1} \int_0^{\pi} (\Delta^2 \phi_{0m} \bar{\phi}_{0m})_{\text{H}} \sin \theta d\theta. \quad (73)$$

Note that the integration of the second term in Eq. (71) is canceled out, because ϕ_{0m} is continuous even at $\theta = \pi/2$. The result given by Eq. (73) shows that the energy extraction from the black hole occurs for incident waves with the frequency parameter σ in the range $0 < |\omega_m| < |m|\Omega_{\text{H}}$ (i.e., $k_m < 0$ for $m < 0$ and $k_m > 0$ for $m > 0$) which means the range $0 < \sigma < \Omega_{\text{H}}$, in accordance with the result of the usual superradiant scattering [9]. We can rewrite the net flux E_m^{H} into the form

$$E_m^{\text{H}} = -\frac{\omega_m}{8k_m Mr_1} \sum_l |C_{lm}^{\text{In}}|^2. \quad (74)$$

in which the coefficient C_{lm}^{In} will be given by solving the radial equation (52) under the low-frequency limit (see Sec. V).

Finally we turn our attention to the radial component of energy flux vector $\langle \mathcal{E}^r \rangle_m$ at infinity. The energy flux vector is written by

$$\langle \mathcal{E}_m^r \rangle_{\infty} = \frac{1}{2\pi r^2} (r^2 \phi_{2m} \bar{\phi}_{2m})_{\infty}. \quad (75)$$

It is easy to see that no ingoing energy flux exists at infinity. Further we calculate the total flux at infinity as follows,

$$E_m^\infty \equiv 2\pi \int_0^\pi \langle \mathcal{E}_m^r \rangle_\infty r^2 \sin \theta d\theta. \quad (76)$$

Then the total flux at infinity is given by

$$E_m^\infty = \int_0^\pi (r^2 \phi_{2m} \bar{\phi}_{2m}) \sin \theta d\theta. \quad (77)$$

Considering the contributions from the Kerr-Schild field and the scattered wave, the total flux at infinity is rewritten by

$$E_m^\infty = \int_0^\pi r^2 [|\phi_{2m}^{\text{KS}}|^2 + 2\text{Re}(\phi_{2m}^{\text{KS}} \bar{\phi}_{2m}^{\text{SW}}) + |\phi_{2m}^{\text{SW}}|^2]_\infty \sin \theta d\theta, \quad (78)$$

where from Eqs. (30c), (35), and (69) we obtain the asymptotic form of Kerr-Schild field as

$$\phi_{2m}^{\text{KS}} \rightarrow -e^{-i\omega_m t + im\varphi + i\omega_m r_*} H_m \frac{a_m e^{a\omega_m \cos \theta} \tan^m(\theta/2)}{\sqrt{2} r r_1 \sin \theta} [(m - a\omega_m) - i\omega_m r_1 \cos \theta], \quad (79)$$

while ϕ_{2m}^{SW} has the asymptotic form given by Eq. (45).

Here, we assume the Kerr-Schild field to be reflection-symmetric with respect to the equatorial plane. This symmetry is corresponding to the condition that the two expansion coefficients are written by

$$a_m = a_{-m}, \quad b_m = -b_{-m}, \quad (80)$$

which allows us to calculate the energy flux for $m < 0$, by using the result for $m > 0$.

In following section, we see the energy transport to calculate the total energy in each region using the low-frequency limit. Then we should solve the radial equation (52) to determine the amplitude of the electromagnetic fields on the horizon and at infinity.

V. LOW-FREQUENCY LIMIT

In the previous section, we discussed the time-averaged energy flux vector on the boundary surface. To evaluate explicitly the efficiency of the superradiant scattering, we attempt to solve the radial equation (52) using the low-frequency limit ($a\omega_m \rightarrow 0$) according to the procedure in [8].

For $a\omega_m \ll 1$, the angular function $S_{lm}^{\pm 1}(\theta)$ is known to be given by

$$\begin{aligned} S_{lm}^{(s)}(\theta) &= (-1)^m \sqrt{\frac{2l+1}{2} \frac{(l+m)! (l-m)!}{(l+s)! (l-s)!}} \sin^{2l}(\theta/2) \\ &\times \sum_{r=0}^{l-s} \left[\binom{l-s}{r} \binom{l+s}{r+s-m} (-1)^{l-r-s} \cot^{2r+s-m}(\theta/2) \right], \end{aligned} \quad (81)$$

where the eigenvalue E is obtained by $E = (l-s)(l+s+1)$ (see [14]). On the other hand, we obtain the radial function through the asymptotic matching method. To consider the dimensionless radial equation from Eq. (52), we introduce the dimensionless condition and parameter as

$$x = \frac{r - r_1}{r_1 - r_2}, \quad (82)$$

$$Q = \frac{r_1^2 + r_2^2}{r_1 - r_2} (m\Omega_H - \omega_m). \quad (83)$$

Then, for $\omega_m M \ll 1$ and $x \gg \max(Q, l)$ (i.e., in the distant region), the radial equation can be rewritten by

$$\frac{d^2 R_{lm}^{(-1)}}{dx^2} + \left[\omega_m^2 (r_1 - r_2)^2 - \frac{2i\omega_m(r_1 - r_2)}{x} - \frac{l(l+1)}{x^2} \right] R_{lm}^{(-1)} = 0. \quad (84)$$

This equation is expressible in terms of the confluent hypergeometric functions, and we obtain the outgoing wave solution given by

$$R_{lm}^{(-1)} = C \frac{2\kappa x e^{-\kappa x}}{(l-1)!} \int_0^\infty e^{-t} t^{l-1} \left(1 + \frac{t}{2\kappa x}\right)^{l+1} dt, \quad (85)$$

with $\kappa = -i(r_1 - r_2)\omega_m$, and $C = F_{lm}^{\text{out}}/(2i\omega_m)$. It is easy to see that the asymptotic behavior of Eq. (85) is written by

$$R_{lm}^{(-1)} \simeq C 2\kappa x e^{-\kappa x}, \quad (86)$$

for $|\kappa x| \gg 1$ in accordance with Eq. (45), and

$$R_{lm}^{(-1)} \simeq C \frac{(2l)!}{(l-1)!} (2\kappa)^{-l-1} x^{-l}, \quad (87)$$

for $|\kappa x| \ll 1$. In the region near the horizon ($x \ll l/\omega_m(r_1 - r_2)$) the radial equation (52) can be rewritten as

$$[x(x+1)]^2 \frac{d^2 R_{lm}^{(-1)}}{dx^2} + [Q^2 - iQ(1+2x) - l(l+1)x(x+1)] R_{lm}^{(-1)} = 0, \quad (88)$$

where it is obtained by neglecting in Eq. (52) all the terms containing ω_m except the one which enters into Q , and the solution with ingoing and outgoing boundary condition at the horizon is given by

$$R_{lm}^{(-1)} = R_{lm}^{(-1)}(\text{In}) + R_{lm}^{(-1)}(\text{Out}), \quad (89)$$

where $R_{lm}^{(-1)}(\text{In})$ and $R_{lm}^{(-1)}(\text{Out})$ are written by

$$\begin{aligned} R_{lm}^{(1)}(\text{In}) &= -A \frac{(2l+1)!}{(l-1)!} \frac{\Gamma(-2iQ)}{\Gamma(2+2iQ)} \frac{\Gamma(1+2iQ)}{\Gamma(l+1-2iQ)} \left(\frac{x}{x+1}\right)^{iQ} x(x+1) \\ &\quad \times \frac{\Gamma(2+2iQ)}{\Gamma(1-l)(l+1)!} \sum_{n=0}^{l-2} \frac{\Gamma(n+1-l)\Gamma(n+l-1)}{\Gamma(n+2+2iQ)} \frac{(-x)^n}{n!}, \end{aligned} \quad (90)$$

$$\begin{aligned} R_{lm}^{(-1)}(\text{Out}) &= A \frac{(2l+1)!}{(l+1)!} \frac{\Gamma(1+2iQ)}{\Gamma(l+1+2iQ)} \left(\frac{x}{x+1}\right)^{iQ} \\ &\quad \times \frac{\Gamma(2iQ)}{\Gamma(-1-l)(l-1)!} \sum_{n=0}^l \frac{\Gamma(n-1-l)\Gamma(n+l)}{\Gamma(n+2iQ)} \frac{(-x)^n}{n!}, \end{aligned} \quad (91)$$

respectively. From Eqs. (90) and (91), the amplitudes D_{lm}^{In} and D_{lm}^{Out} are given by

$$D_{lm}^{\text{In}} = -(r_1 - r_2)^2 A \frac{(2l+1)!}{(l-1)!} \frac{\Gamma(-2iQ)}{\Gamma(2+2iQ)} \frac{\Gamma(1+2iQ)}{\Gamma(l+1-2iQ)}, \quad (92)$$

$$D_{lm}^{\text{Out}} = A \frac{(2l+1)!}{(l+1)!} \frac{\Gamma(1+2iQ)}{\Gamma(l+1+2iQ)}. \quad (93)$$

We obtain the solution with the asymptotic behavior as follows,

$$R_{lm}^{(-1)} \simeq (-1)^{-l} A x^{-l}, \quad (94)$$

for $x \gg 1$. If all the parameters satisfy the condition $\max(Q, l) \ll l/\omega_m(r_1 - r_2)$, we have the overlap region in which both the outer expression (87) and the inner expression (94) hold. In this region, we can match the leading-terms of the solutions (87) and (94), and this matching yields

$$A = (-1)^l (2\kappa)^{-l-1} \frac{(2l)!}{(l+s)!} C. \quad (95)$$

Then we can obtain the ratios $F_{lm}^{\text{Out}}/D_{lm}^{\text{Out}}$ and $D_{lm}^{\text{In}}/D_{lm}^{\text{Out}}$ as follows,

$$\frac{F_{lm}^{\text{Out}}}{D_{lm}^{\text{Out}}} = (-1)^{-l}(r_1 - r_2)^{-1}(2\kappa)^{l+1} \frac{(l-1)!(l+1)!}{(2l)!(2l+1)!} \frac{\Gamma(l+1+2iQ)}{\Gamma(1+2iQ)}, \quad (96)$$

$$\frac{D_{lm}^{\text{In}}}{D_{lm}^{\text{Out}}} = -\frac{1}{(r_1 - r_2)^2} \frac{(l+1)!}{(l-1)!} \frac{\Gamma(l+1+2iQ)\Gamma(-2iQ)}{\Gamma(l+1-2iQ)\Gamma(2+2iQ)}. \quad (97)$$

Here the coefficients F_{lm}^{Out} and D_{lm}^{In} represent the amplitude of the outgoing waves at infinity and the ingoing waves on the horizon in the component ϕ_2^{SW} , respectively. Further, from Eqs. (46) and (97) it is easy to see the coefficient C_{lm}^{In} describing the ingoing waves in the component ϕ_0^{SW} to be

$$\frac{C_{lm}^{\text{In}}}{D_{lm}^{\text{Out}}} = 2 \frac{\Gamma(l+1+2iQ)}{\Gamma(l+1-2iQ)}. \quad (98)$$

Note that from Eq. (98) the ratio $|C_{lm}^{\text{In}}|^2/|D_{lm}^{\text{Out}}|^2$ is given by

$$|C_{lm}^{\text{In}}|^2/|D_{lm}^{\text{Out}}|^2 = 4. \quad (99)$$

Next we pay attention to the energy extraction from the black hole. From Eq. (71), the total flux on the horizon is given by the component ϕ_0 . On the other hand, the scattered field is induced and connected by the Kerr-Schild field on the horizon through Eq. (51). Further, using Eq. (69) which requires the absence of the disk on the horizon, the coefficient D_{lm}^{Out} is obtained as

$$D_{lm}^{\text{Out}} = -ia_m \frac{ak_m}{r_1 \Omega_H}. \quad (100)$$

Therefore, using the ratio $C_{lm}^{\text{In}}/D_{lm}^{\text{Out}}$ given by Eq. (98) we can evaluate the component ϕ_0 near the horizon as

$$\phi_{0m} \simeq \sum_{l \geq |m|} e^{-im\omega_m t + im\varphi} S_{lm}^{(1)} \Delta^{-1} e^{-ikr_*} 2 \frac{\Gamma(l+1+2iQ)}{\Gamma(l+1-2iQ)} D_{lm}^{\text{Out}}, \quad (101)$$

which is useful to see the distribution of the energy flux on the horizon through Eq. (71).

The θ -dependence of $\Delta^2|\phi_{0m}|^2$ on the horizon is shown in Fig. 1, from which we can see that a highly asymmetric

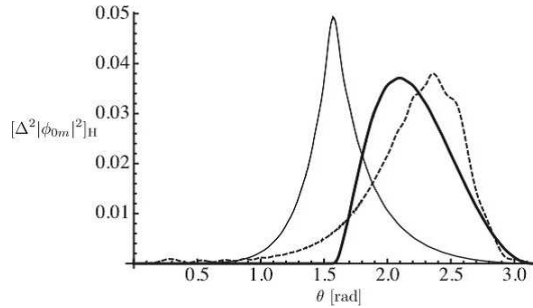


FIG. 1: The amplitude $\Delta^2|\phi_{0m}|^2$ of the mode with $m = 2$ on the horizon as a function of zenithal angle θ . The thin line, dashed line and heavy line correspond to the cases $Q = 0$ ($a = 0$), $Q = 10$ ($a \simeq 0.995$), and $Q \rightarrow \infty$ ($a \rightarrow 1$), respectively.

profile appears as the spin parameter a increases. For example, for the modes with $m \geq 2$ the peak of $\Delta^2|\phi_{0m}|^2$ exists in the lower hemisphere $\pi/2 < \theta < \pi$, though the K-S field ϕ_{2m}^{KS} vanishes in this range. On the contrary, for the modes with $m \leq -2$, the peak of $\Delta^2|\phi_{0m}|^2$ on the horizon exists in the upper hemisphere $0 < \theta < \pi/2$ according to the relation $\Delta^2|\phi_{0m}(\theta)|^2 = \Delta^2|\phi_{0-m}(\pi - \theta/2)|^2$. Even if the net energy flux estimated by the sum $\langle \mathcal{E}^r(\theta) \rangle_H \equiv \sum_{m \geq 2} \langle \mathcal{E}_m^r(\theta) \rangle_H + \sum_{m \leq -2} \langle \mathcal{E}_m^r(\theta) \rangle_H$ has a θ -dependence symmetric with respect to the equatorial plane, the asymmetric profile of $\sum_{m \geq 2} \langle \mathcal{E}_m^r(\theta) \rangle_H$ given by $\Delta^2|\phi_{0m}(\theta)|^2$ is an interesting feature of the Kerr black hole.

Now let us evaluate E_m^{D} in Eq. (66) and E_m^{∞} in Eq. (76) under the low-frequency approximation, keeping the terms up to the first order in $M\omega_m$. The total flux radiated from the disk surface is given as

$$E_m^{\text{D}} \simeq \frac{(m - a\omega_m)}{4r_1} |a_m|^2, \quad (102)$$

while the total flux at infinity is obtained by

$$E_m^\infty \simeq \frac{(m - 2a\omega_m)|a_m|^2}{4r_1^2}. \quad (103)$$

The difference

$$E_m^D - E_m^\infty \simeq \frac{a\omega_m}{2r_1^2}|a_m|^2, \quad (104)$$

means the energy inflow from the disk to the black hole. On the other hand, from Eqs. (74), (101) and (99) the net extracted energy on the horizon is calculated as

$$E_m^H \simeq \frac{a\omega_m}{4r_1^2}|a_m|^2. \quad (105)$$

Now we can discuss the energy transported in disk-black hole system under the low-frequent approximation. From Eq. (104) a part of disk emission turns out to be transported to the black hole. On the other hand, from Eq. (105), it is easy to see that the net flux on the horizon indicates the energy extraction from the black hole. Considering the energy conservation in the disk-black hole system, the energy flow (104) transported from disk to black hole induces the black hole superradiance (105), and the total flux $E_m^H + (E_m^D - E_m^\infty)$ returns to the disk surface (see Fig. 2). Then,

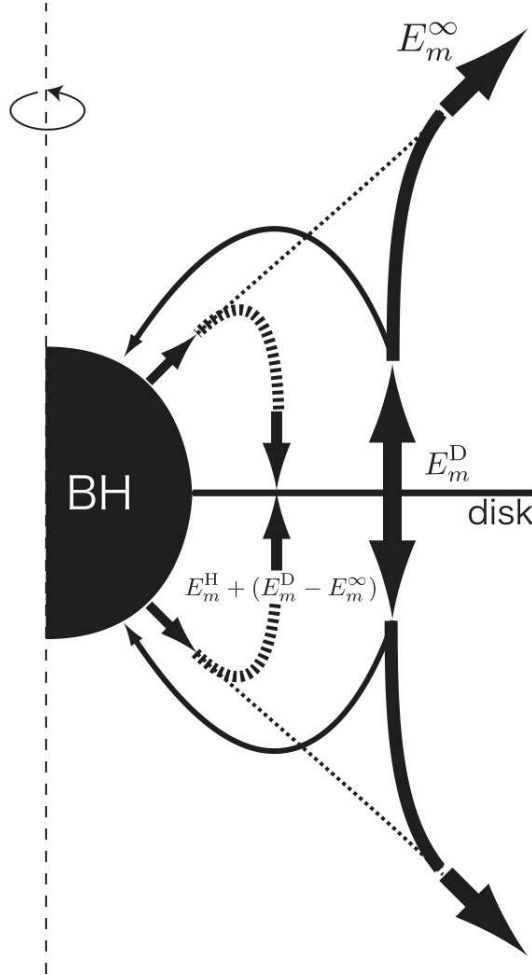


FIG. 2: Schematic diagram of energy transport in the disk-black hole system. The energy radiated from disk surface is shown as solid arrows. The superradiant energy flows from the black hole to the disk surface and to infinity are shown as heavy dashed and thin dashed arrows, respectively.

it is interesting to note that the feedback energy flow is estimated to be

$$E_m^H + (E_m^D - E_m^\infty) = \frac{3}{2}(E_m^D - E_m^\infty), \quad (106)$$

which is one and half times as large as the energy flow $E_m^D - E_m^\infty$ from disk to black hole. On the other hand, if we consider the limit $a \rightarrow 0$ (i.e. the case of the Schwarzschild black hole), the net flux on the horizon is given by

$$E_m^H \simeq -\frac{\omega_m^2}{4m} |a_m|^2, \quad (107)$$

that is, no energy feedback from the black hole to the disk occurs.

Finally let us discuss the energy flux observed at infinity. From Eq. (78), the net flux at infinity can be divided as follows,

$$E_m^\infty = F_m^D + F_m^{D-H} + F_m^H, \quad (108)$$

where

$$F_m^D = \int_0^\pi H_m^2 |a_m|^2 \frac{e^{2a\omega_m \cos \theta} \tan^{2m}(\theta/2)}{2r_1^2 \sin \theta} [(m - a\omega_m)^2 + \omega_m^2 r_1^2 \cos^2 \theta] d\theta, \quad (109)$$

$$F_m^{D-H} = 2\text{Re} \left\{ \int_0^\pi H_m \frac{a_m e^{a\omega_m \cos \theta} \tan^m(\theta/2)}{\sqrt{2}r_1} [(m - a\omega_m) - i\omega_m r_1 \cos \theta] \sum_l \bar{F}_{lm}^{\text{Out}} S_{lm}^{(-1)}(\theta) d\theta \right\}, \quad (110)$$

$$F_m^H = \sum_{l'} \int_0^\pi F_{lm}^{\text{Out}} \bar{F}_{l'm}^{\text{Out}} S_{lm}^{(-1)}(\theta) S_{l'm}^{(-1)}(\theta) \sin \theta d\theta. \quad (111)$$

Here from Eqs. (96) and (100) the coefficient F_{lm}^{Out} is obtained as

$$F_{lm}^{\text{Out}} = (-1)^{-l+1} a_m \frac{(l-1)!(l+1)!}{(2l)!(2l+1)!} \frac{\Gamma(l+1+2iQ)}{\Gamma(l+2iQ)} \frac{2^{l+1}(r_1-r_2)^l a \omega_m^{l+1} k_m}{r_1 \Omega_H}. \quad (112)$$

We interpret F_m^D , F_m^H and F_m^{D-H} as the direct radiation from the disk, the net flux of the scattered wave caused by superradiant scattering, and their interference effect, respectively. To find the contribution of the scattered radiation in the net flux at infinity (108), we pay attention to the dependence of frequency ω_m . From Eqs. (100), (108), (110) and (112), it is easy to see that in the interference effect F_m^{D-H} , the scattered radiation appear from the order of $(M\omega_m)^{l+1}$. However, this term is very smaller than the disk radiation term with the order of $(M\omega_m)^0$ in Eq. (109), if the low-frequency limit is considered.

From the viewpoint of highly energetic astrophysical phenomena, the superradiant transport of the energy flux to the disk will be interesting as a black hole feedback mechanism which plays a role of disk heating, while the contribution of the superradiant scattering of waves to the energy flux at infinity will be useful for an observational check of the black hole spin (see Fig. 2). Unfortunately, as was above-mentioned, the superradiant part at infinity remains much smaller than the direct $M\omega_m = Mm\sigma \ll 1$ is used. Therefore, it is important to analyze the case such that $M\sigma \sim 1$ for the frequency parameter σ describing the disk radiation, by keeping the superradiance condition $\sigma < \Omega_H$ and the regularity condition $a\sigma < 1$ for the Kerr-Schild field in any region except on the disk.

For high m modes giving $m > 1/M\sigma$ the superradiant effect may be also suppressed. Nevertheless, we must remark that the contribution of such modes should be taken into account if one estimates the total flux of the disk radiation given by the function $\psi(X)$ and $\Psi(X)$ singular at the equatorial plane. This is because as a result of the existence of the branch point at $|X| = 1$ the coefficients a_m and b_m in the expansion form (34) and (35) do not rapidly decrease as m increase. Then, the high m modes of the vacuum field given by ϕ_0^{SW} and ϕ_2^{SW} should be also efficiently generated from the disk radiation corresponding to the Kerr-Schild m modes. Even if the superradiant effect is small for each high m mode, the total sum (61) of the energy fluxes $\langle \mathcal{E}_m^\mu \rangle$ for all m modes will be an important task to discuss more clearly the energy transport in the disk-black hole system. Because our main purpose in this paper is focused on the construction of the basic formulae to calculate of the energy fluxes at the horizon, the equatorial disk and the far distant region, such a calculation of the total energy flux will be investigated in future works.

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